Math 214 - Final
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Fall 2017 - Dr. Richard Aoun

Exercise 1. Let X and Y be two topological spaces and $f : X \longrightarrow Y$ a map. Show that if X is compact and f is continuous on X, then $f(X)$ is compact (when endowed with the subspace topology).

Exercise 2. **Parts 1,2 and 3 are mutually independent**.
We endow $\mathbb{R}$ and $\mathbb{R}^2$ with the Euclidean topology and their respective subsets with the subspace topology. Recall that $\mathbf{S}^1$ denotes the unit circle in $\mathbb{R}^2$, i.e. the sphere of center the origin and radius 1 for the Euclidean metric.

1. Let X and Y be two topological spaces. Assume X to be compact and Y to be Hausdorff.
   (a) Show that any continuous map $f : X \longrightarrow Y$ is a closed map.
   (b) Deduce that any continuous injection from X into Y is an embedding, i.e. yields a homeomorphism between X and $f(X)$.
2. Let $f : \mathbf{S}^1 \longrightarrow \mathbb{R}$ be a continuous map. Explain why there exists two real numbers $a \leq b$ such that $f(\mathbf{S}^1) = [a, b]$.
3. Show that there is no homeomorphism between $\mathbf{S}^1$ and any closed and bounded interval $[a, b]$.
4. **Deduce** from the previous question**S** that there is no continuous injection from $\mathbf{S}^1$ into $\mathbb{R}$.

Exercise 3. Let (X, d) be a metric space.

1. Show that a Cauchy sequence in (X, d) that admits a convergent subsequence is actually convergent.
2. Suppose that there exists some $r_0 > 0$ such that every open ball in (X, d) of radius r_0 has compact closure. Show that (X, d) is complete.
3. Suppose that for each $x \in X$, there is an $r = r(x) > 0$ such that the open ball $B(x, r)$ has compact closure. Show by means of an example that X need not be complete.
   Check very quickly the properties you are claiming about your spaces.

Exercise 4. *In this exercise we adopt the following definition of countability: A countable set is a set that is in bijection with a subset of $\mathbb{N}$. Equivalently, it is either a finite set or an infinite one but in bijection with $\mathbb{N}$. You can use without any proof that a countable union (in particular a finite one) of countable sets is still countable.*

Prove that any compact metric space is separable, i.e. contains a countable dense subset.

Exercise 5. (Arzelà-Ascoli Theorem)
Our goal is to prove and discuss a weak form of *Arzelà-Ascoli's Theorem*, which is an important result in functional analysis. Our setting is a compact metric space (X, d_X). Denote by $\mathcal{C}(X)$ the set of all continuous functions from X to $\mathbb{R}$. We consider the sup-norm

$$d_\infty(f, g) = \sup\{|f(x) - g(x)|; x \in X\}.$$

The aforementioned theorem (see Question 5 for the statement) gives a criteria for a subset of $\mathcal{C}(X)$ to be compact. Other than boundness and being closed, an additional assumption should be imposed: that of *equicontinuity* of a family of continuous functions on X.

1. Explain why d_∞ is well-defined and that the "sup" in its definition can be replaced by a "max". No need to check that it is indeed a metric on $\mathcal{C}(X)$.

2. **Definition:** A subset $\mathcal{F} \subseteq \mathcal{C}(X)$ is said to be *equicontinuous* if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for every $f \in \mathcal{F}$ and for every $x, y \in X$,

$$d_X(x, y) < \delta \Longrightarrow |f(x) - f(y)| < \epsilon.$$

   (a) Explain why if $\mathcal{F}$ is a singleton, i.e. $\mathcal{F} = \{f_0\}$ for some continuous function $f_0 : X \longrightarrow \mathbb{R}$, then $\mathcal{F}$ is equicontinuous.

   In the questions (b), (c) and (d) below, we take $X = [0, 1]$.

   (b) Check that, for every $C > 0$, the set $\mathcal{F}_C$ of differentiable functions f on $[0, 1]$ such that $\sup_{0 \le x \le 1} |f'(x)| \le C$ is equicontinuous.

   (c) Show that $\mathcal{F} := \{x \mapsto \sin(nx); n \in \mathbb{N}\}$ is a subset of $\mathcal{C}([0, 1])$ which is not equicontinuous. Is the set $\bigcup_{C>0} \mathcal{F}_C$ of differentiable functions on $[0, 1]$ with bounded derivative equicontinuous?
   Beside a rigorous proof, draw a picture and explain briefly what is happening.

   (d) Denote by 0 the constant function $x \mapsto 0$ on $[0, 1]$. Show that the closed ball of center 0 and radius 1 in $(\mathcal{C}([0, 1], d_\infty)$ is not equicontinuous.
   Beside a rigorous proof, draw a picture and explain briefly what is happening.

3. Show that, if a subset $\mathcal{F}$ of $\mathcal{C}(X)$ is compact, then $\mathcal{F}$ is necessarily equicontinuous.

4. ** Let $\mathcal{F} \subseteq \mathcal{C}(X)$. Show that if $\mathcal{F}$ is **bounded** for the metric d_∞ and **equicontinuous**, then $\mathcal{F}$ is totally bounded[1].

5. We admit the following criteria of compactness: *"a metric space is compact if and only if it is totally bounded and complete"*. Deduce from the previous questions Arzelà-Ascoli's theorem:

$$\mathcal{F} \subset \mathcal{C}(X) \text{ is compact} \iff \mathcal{F} \text{ is closed, bounded and equicontinuous.}$$

6. BONUS; BUT WILL BE COUNTED ONLY IF QUESTION 4 IS TREATED.
   Let $n \in \mathbb{N}^*$. Show that Heine-Borel's theorem in $\mathbb{R}^n$ can be deduce from Arzelà-Ascoli's theorem, by choosing a suitable compact metric space X.

[1] Recall that if A is a subset of a metric space (Y, ρ), then A is said to be totally bounded, if for every $\epsilon > 0$ one can find finitely many points $a_1, \cdots, a_r \in A$ (that depend on ϵ) such that $A \subseteq \bigcup_{i=1}^r B(a_i, \epsilon)$.